Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

FUNCTIONS OF LIMITED VARIATION IN AN INFINITE NUMBER OF DIMENSIONS.

BY P. J. DANIELL.

In a recent issue of these Annals appeared a paper by the author on "A General Form of Integral."* Integration was defined for functions of perfectly general elements, p. In a later paper (in the last issue), the author was able to define two kinds of integral in a space of a denumerably infinite number of dimensions. One of these was the generalization of the Lebesgue integral in the interval 0 to 1, the other an iterated Stieltjes integral of positive type. In the present paper functions, of a more general class, which are of limited variation in a denumerably infinite number of dimensions, are defined together with their corresponding Stieltjes integrals. References are made to Fréchet's thesis.† Our attention will be confined in this paper to a "finite domain" or interval

$$a_1 \leqq x_1 \leqq b_1, \ \cdots, \ a_n \leqq x_n \leqq b_n, \ \cdots.$$

1. **Introductory remarks.** For the sake of definiteness we first define what we mean by a function of limited variation in a finite number of dimensions. Let $\alpha(x_1, x_2, \cdots x_n)$ be a function of a finite number of variables in the finite domain

$$(a_1 \leqq x_1 \leqq b_1, \ \cdots, \ a_n \leqq x_n \leqq b_n).$$

Divide the ranges of the variables into subranges as, for example, by the numbers

$$\begin{array}{c} a_1 = x_{10} < x_{11} < \cdots < x_{1A} = b_1 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_n = x_{n0} < x_{n1} < \cdots < x_{nk} = b_n. \end{array}$$

This divides the original interval into a finite number of rectangles or subintervals which we name by their extreme upper corners. We denote

$$\begin{aligned} \Delta_i\alpha &= \alpha(x_{1i}, x_2, x_3 \cdots x_n) - \alpha(x_{1i-1}, x_2, \cdots, x_n), \\ \Delta_{ij}{}^2\alpha &= \Delta_j(\Delta_i\alpha) = \alpha(x_{1i}, x_{2j}, x_3, \cdots, x_n) \\ &\qquad - \alpha(x_{1i-1}, x_{2j}, x_3 \cdots x_n) - \alpha(x_{1i}, x_{2j-1}, x_3, \cdots, x_n) \\ &\qquad\qquad + \alpha(x_{1i-1}, x_{2j-1}, x_3, \cdots, x_n), \end{aligned}$$

---

* P. J. Daniell, These Annals, vol. 19 (1918), p. 279. In this paper the author erroneously stated that E. H. Moore's integral is a special instance of his. He takes this opportunity to offer his apologies to Professor Moore and to his readers.

† M. Fréchet, Rendiconti di Circolo matematico di Palermo, vol. 22 (1906), p. 40.

and so on; finally

$$\Delta_{ij}{}^{n}{}_{\cdots q}\alpha = \alpha(x_{1i}, x_{2j}, \cdots, x_{nq}) - \cdots + \cdots$$
$$\cdots + (-1)^{n}\alpha(x_{1i-1}, x_{2j-1}, \cdots, x_{nq-1}).$$

DEFINITION 1. 1. *Positive type.* The function $\alpha(x_1, \cdots, x_n)$ is said to be of positive type in the variables $(x_1, \cdots, x_n)$, if

$$\Delta_{ij}{}^{n}{}_{\cdots q}\alpha \geq 0$$

for all subintervals $(i, j, \cdots, q)$ whatever.

DEFINITION 1. 2. *Limited variation.* The function $\alpha(x_1, \cdots, x_n)$ is said to be of limited variation in the variables $(x_1, \cdots, x_n)$ if

$$\sum_{ij\ldots q} | \Delta_{ij}{}^{n}{}_{\cdots q}\alpha | \leq K,$$

where K is some finite number independent of the manner in which the interval is divided. The lower bound of such numbers K is called the total variation of α in the interval $(a_1, a_2, \cdots, a_n)$ to $(b_1, b_2, \cdots, b_n)$.

DEFINITION 1. 3. *Simple type.* The function $\alpha(x_1, \cdots, x_n)$ is said to be of simple type in the interval $(a_1, a_2, \cdots, a_n)$ to $(b_1, b_2, \cdots, b_n)$, if it is zero whenever any one of the variables, x_r, attains its lower limit, a_r.

THEOREM 1. 4. *A function which is of positive and simple type in the variables $(x_1, \cdots, x_n)$ is also of positive type in any collection of variables chosen from among $(x_1, \cdots, x_n)$, the remainder being held constant.*

Let $x_p, x_q, \cdots, x_t$, be the m variables chosen and consider an interval $(c_p, c_q, \cdots, c_t)$ to $(x_p, x_q, \cdots, x_t)$. The nth order difference of α in the interval

$$(a_1, a_2, \cdots c_p, \cdots, c_t, \cdots a_n) \text{ to } (x_1, \cdots, x_n),$$

that is to say where $a_1, \cdots, a_n$ are the lower limits for all variables except the $p, q, \cdots t$ variables,

$$\Delta\alpha^{n} = \alpha(x_1, \cdots, x_n) - \cdots + \cdots + (-1)^{n}\alpha(a_1, \cdots, c_p, \cdots, a_n)$$
$$= \alpha(x_1, \cdots, x_n) - \cdots + \cdots + (-1)^{m}\alpha(x_1, \cdots, c_p, \cdots, c_t, \cdots, x_n).$$

For, since α is of simple type, the remaining values of α all vanish.

[Example. In the interval (a_1, c_2) to (x_1, x_2)

$$\Delta^2\alpha(x_1x_2) = \alpha(x_1, x_2) - \alpha(x_1, c_2) - \alpha(a_1, x_2) + \alpha(a_1, c_2)$$
$$= \alpha(x_1, x_2) - \alpha(x_1, c_2) - 0 + 0$$
$$= \Delta\alpha(x_1, x_2)$$

in the interval c_2 to x_2, x_1 being constant.]

Then the mth order difference of α in the interval $(c_p, c_q, \cdots c_t)$ to $(x_p, x_q, \cdots x_t)$, the other variables $x_1, \cdots, x_n$ being held constant, is equal to the nth order difference of α in the interval

$$(a_1, \cdots, c_p, \cdots, c_t, \cdots, a_n) \text{ to } (x_1, \cdots, x_n).$$

If α is of positive type, the latter nth order difference is non-negative, or the former mth order difference is also non-negative. The theorem is proved.

Theorem 1. 5. *If $\alpha(x_1, \cdots, x_n)$ is of limited variation and simple type in the variables $(x_1, \cdots, x_n)$, it is also of limited variation in any collection of variables chosen from among $(x_1, \cdots, x_n)$, the remainder being held constant. Also, if the* nth *order total variation $\leqq K$, the latter variation is also $\leqq K$.*

Let the interval $(a_p, a_q, \cdots, a_t)$ to $(b_p, b_q, \cdots, b_t)$ be subdivided into subintervals in any manner, one of these subintervals being

$$(c_p, c_q, \cdots, c_t) \text{ to } (d_p, d_q, \cdots, d_t).$$

By the reasoning employed in Theorem 1. 4, the multiple order difference of α in this interval is the same as the nth order difference of α in the interval

$$(a_1, \cdots, c_p, \cdots, c_t, \cdots, a_n) \text{ to } (x_1, \cdots, d_p, \cdots, d_t, \cdots, x_n).$$

The collection of subintervals of the latter type form a possible mode of subdivision of the n-dimensional interval and, therefore,

$$\Sigma \mid \Delta^m \alpha \mid \leqq K,$$

if K is the total variation of α in the variables $(x_1, \cdots, x_n)$, in the interval $(a_1, \cdots, a_n)$ to $(b_1, \cdots, b_n)$. This proves the theorem.

[Example. Consider the variation of $\alpha(\xi_1, x_2)$ in the interval, $a_1 \leqq \xi_1 \leqq b_1$. Divide this interval by the numbers

$$a_1 = l_0 < l_1 < \cdots < l_n = b_1,$$

$$\alpha(l_r, x_2) - \alpha(l_{r-1}, x_2) = \alpha(l_r, x_2) - \alpha(l_r, a_2) - \alpha(l_{r-1}, x_2) + \alpha(l_{r-1}, a_2).$$

$$\sum_{r=1}^{n} \mid \Delta_r \alpha \mid = \sum_{r=1}^{n} \mid \Delta^2_{r, x_2} \cdot \alpha \mid$$

$$\leqq K.]$$

2. Infinite number of dimensions.

Definition 2. 1. *Positive type.* A function $\alpha(x_1, \cdots, x_n, \cdots)$ of the denumerably infinite number of variables $(x_1, x_2, \cdots, x_n, \cdots)$ is said to be of positive type, if, when $x_{n+1}, x_{n+2}, \cdots$ are held constant, it is of positive type in $(x_1, \cdots, x_n)$ for all integers n.

DEFINITION 2. 2. *Limited variation.* A function $\alpha(x_1, \cdots, x_n, \cdots)$ of the denumerably infinite number of variables $(x_1, \cdots x_n, \cdots)$ is said to be of limited variation, if, when $x_{n+1}, x_{n+2}, \cdots$ are held constant, it is of limited variation in $(x_1, \cdots, x_n)$ for all integers n, *these total variations being limited in their set.*

DEFINITION 2. 3. *Simple type.* A function $\alpha(x_1, \cdots, x_n, \cdots)$ is said to be of simple type in the interval

$$a_1 \leqq x_1 \leqq b_1, \cdots, a_n \leqq x_n \leqq b_n, \cdots,$$

if it is zero whenever any one of the variables, x_n, attains its lower limit, a_n.

THEOREM 2. 4. *Any function of limited variation and simple type is the difference of two limited functions of simple and positive type. Conversely the difference of two limited functions of simple and positive type is a function of limited variation and simple type.*

The second part of the theorem is easy to prove; for, if $\alpha = \beta - \beta'$,

$$\Delta^n\alpha = \Delta^n\beta - \Delta^n\beta',$$

$$|\Delta^n\alpha| \leqq \Delta^n\beta + \Delta^n\beta'$$

$$\Sigma\,|\Delta^n\alpha| \leqq \beta(b_1, b_2, \cdots, b_n, \cdots) + \beta'(b_1, b_2, \cdots, b_n, \cdots).$$

Also when $\beta = 0$, $\beta' = 0$, then $\alpha = 0$.

The proof of the first part is not so simple. Let K denote the upper bound of all variations of α, varying $(x_1, \cdots, x_n)$, holding $x_{n+1}, \cdots$, constant for all integers n and all $(x_{n+1}, \cdots)$ in the interval. By definition such a finite K exists. Denote by

$$\alpha_n(x_1, \cdots, x_n, x_{n+1}, \cdots)$$

the total variation of $\alpha(\xi_1, \xi_2, \cdots \xi_n, x_{n+1}, \cdots)$ holding $(x_{n+1}, x_{n+2}, \cdots)$ constant and varying $(\xi_1, \cdots, \xi_n)$ in the interval $(a_1, a_2, \cdots, a_n)$ to $(x_1, \cdots, x_n)$. This function is numerically not greater than K and is of simple type in all the variables. It is also of positive type in the variables $(x_1, \cdots, x_n)$, when $x_{n+1}, \cdots$, are regarded as parametric constants. By Theorem 1. 4. if $m \leqq n$, α_n is of positive type in the variables $(x_1, \cdots, x_m)$. Let

$$2\beta_n = \alpha + \alpha_n, \qquad 2\beta_n' = \alpha_n - \alpha.$$

Then β_n, β_n' are also not greater than K and are of simple type in all the variables. They are also of positive type in the variables $(x_1, \cdots, x_n)$, in fact they are the positive and negative variation functions of α in the n variables such that

$$\beta_n - \beta_n' = \alpha,$$

$$\beta_n + \beta_n' = \alpha_n,$$

the total variation function of α in the variables $(x_1, \cdots, x_n)$. By Theorem 1. 5 if $m \leqq n$,

$$\alpha_m \leqq \alpha_n.$$

Then the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\beta_n'\}$ are nondecreasing sequences limited by K and possess limits ω, β, β', such that

$$\alpha = \beta - \beta', \qquad \beta + \beta' = \omega.$$

DEFINITION 2. 5. We define the total variation function,

$$\omega(x_1, x_2, \cdots, x_n, \cdots) = \lim_{n=\infty} \alpha_n(x_1, \cdots, x_n, \cdots),$$

as the limit, as n increases indefinitely, of the total variation function, α_n, of α in the variables $(x_1, \cdots, x_n)$, holding $x_{n+1}, \cdots$ constant. Similarly we can call β, β', the positive and negative variation functions respectively. Since β_n, β_n' are of positive and simple type in $(x_1, \cdots, x_n)$, by Theorem 1. 4 if $m \leqq n$, they are also of positive type in the variables $(x_1, \cdots, x_m)$. Therefore, in the limit, β, β' are of positive type in the variables $(x_1, \cdots, x_m)$ for all integers m; that is to say β, β' are of positive type in the sense of Def. 2. 1. Again if $x_r = a_r$, $\beta_n = 0$, $\beta_n' = 0$ for all n, or

$$\beta = 0, \qquad \beta' = 0.$$

Hence β, β' are of simple type, and they are not greater than K. The theorem is proved.

3. **Generalized Stieltjes integral.** Let $\beta\ (x_1, \cdots, x_n, \cdots)$ be a limited function of simple and positive type in the interval

$$(a_1, \cdots, a_n, \cdots) \text{ to } (b_1, \cdots, b_n, \cdots).$$

as in our previous paper on integrals in an infinite number of dimensions, we choose the class T_0 to be the class of functions of a finite number of the variables, continuous in the given interval. As before T_0 is closed with respect to the operations of multiplication by a constant, addition and taking the modulus; the functions are bounded, and if $f_1(p), \cdots, f_n(p), \cdots$ are functions of class T_0 such that

$$f_1(p) \geqq f_2(p) \geqq \cdots \geqq f_n(p) \geqq \cdots \geqq 0 = \lim_{n=\infty} f_n(p)$$

for all p in the interval; then

$$\lim_{n=\infty} \max_p f_n(p) = 0.$$

DEFINITION 3. 1. Let $f(x_p, \cdots, x_t)$ be a function of class T_0, then we define the integral of f with respect to β in the given interval as

$$\int f d\beta = \int_{a_p}^{b_p} \cdots \int_{a_t}^{b_t} f(x_p, \cdots x_t) dB(x_p, \cdots x_t),$$

where the integration is over the finite number of variables $(x_p, \cdots, x_t)$ and where

$$B(x_p, \cdots, x_t) = \beta(b_1, \cdots, x_p, \cdots, x_t, \cdots, b_n),$$

all the variables except $(x_p, \cdots, x_t)$ being made equal to their upper bounds $(b_1, \cdots, b_n, \cdots)$. The right-hand integral is definite because by Theorem 2. 4, and Definition 2. 1, $B(x_p, \cdots, x_t)$ is a limited function of positive type and f is a continuous function of the finite number of variables $(x_p, \cdots, x_t)$.

It happens that a continuous function of m variables can also be regarded as a continuous function of $m+1$, $m+2$, $\cdots$ variables, by subjoining variables with respect to which the function is constant. On this account it is necessary to prove that our definition of the integral is self-consistent. It is sufficient if we consider the case where one variable is added; that is, if x_u is any variable other than $x_p, x_q, \cdots, x_t$, then we can prove that, if $f(x_p, \cdots, x_t)$ is a continuous function,

$$\int f(x_p, \cdots, x_t)dB(x_p, \cdots, x_t) = \int f(x_p, \cdots, x_t)dB(x_p, \cdots, x_t, x_u).$$

[For convenience we have placed the variable x_n last, but it may occur anywhere with respect to $x_p, \cdots, x_t$.] The right-hand integral is found as the limit of

$$\sum_{a, \ldots, \theta, \eta} f(\xi_{pa}, \cdots, \xi_{t\theta})\Delta_{x_{pa} \cdots x_{t\theta} x_{u\eta}}^{x_{p, a+1} \cdots x_{t, \theta+1} x_{u, \eta+1}} B(x_p \cdots x_t, x_u)$$
$$= \sum_{a, \ldots, \theta} f(\xi_{pa}, \cdots, \xi_{t\theta})\Delta_{x_{pa} \cdots x_{t\theta}}^{x_{p, a+1} \cdots x_{t, \theta+1}} \sum \Delta_{x_{u\eta}}^{x_{u, \eta+1}} B(x_p \cdots x_t x_u).$$

But

$$\sum_{\eta} \Delta_{x_{u\eta}}^{x_{u, \eta+1}} B(x_p, \cdots, x_t x_u) = B(x_p, \cdots, x_t, b_u) - B(x_p, \cdots, x_t, a_u)$$
$$= B(x_p, \cdots, x_t) - 0.$$

This proves the consistency of the integral definition. It follows without further difficulty that the integral

$$I(f) = \int f d\beta$$

satisfies the postulates,

(C) $$I(cf) = cI(f),$$

(A) $$I(f_1 + f_2) = I(f_1) + I(f_2),$$

(P) $$I(f) \geqq 0, \text{ if } f \geqq 0 \text{ for all elements } p.$$

Also if

$$B = \beta(b_1, \cdots, b_n, \cdots)$$

$$| I(f) | \leqq B \max_p | f(p) |.$$

Using the fact stated in the paragraph before Definition 3. 1, we observe that postulate (L) is also satisfied, that is to say, if

$$f_1(p) \geq f_2(p) \geq \cdots \geq f_n(p) \cdots \geq 0 = \lim f_n(p)$$

for all elements p, then

$$\begin{aligned} \lim_{n=\infty} I(f_n) &\leq B \lim_{n=\infty} \max_p f(p) \\ &= 0. \end{aligned}$$

DEFINITION 3. 2. Let $\alpha(x_1, \cdots, x_n, \cdots)$ be a function of limited variation and simple type. By Theorem 2. 4 it is the difference of two functions β, β' limited and of simple and positive type. We define

$$S(f;\, \alpha) = I(f;\, \beta) - I(f;\, \beta'),$$

$$\int f d\alpha = \int f d\beta - \int f d\beta'.$$

$S(f)$ satisfies postulates (C) (A) (L). Moreover

$$|\, S(f;\, \alpha) \,| \leq (B + B') \max |f|,$$

that is to say, it also satisfies postulate M where

$$M(f) = \max f.$$

As we proved in the previous paper any continuous function is the limit of a sequence of functions of class T_0 limited in their set and therefore we can define $S(f;\, \alpha) = \int f d\alpha$ for all continuous functions f. We can also extend the definition of the Stieltjes integral to a wider class of summable functions by the methods used in our article "A General Form of Integral."

4. *Example* 1. Let $\alpha_i(x_i)$, $\alpha_{ij}(x_i, x_j)$, $\cdots$ be functions of simple type and limited variation in the variables specified, in the interval

$$(a_1, \cdots, a_n, \cdots) \text{ to } (b_1, \cdots, b_n, \cdots);$$

and let V_i, V_{ij}, $\cdots$ be the corresponding total variations. Then if

$$\sum_{n=1}^{\infty} |\, k_i \,| V_i = S_1, \qquad \sum_{i,j} |\, k_{ij} \,| V_{ij} = S_2, \cdots$$

are convergent series and if

$$\sum_{n=1}^{\infty} S_n = K$$

is convergent, we may define

$$\alpha(x_1, \cdots, x_n, \cdots) = \sum k_i \alpha_i(x_i) + \sum_{i,j} k_{ij} \alpha_{ij}(x_i, x_j) + \cdots,$$

and this α will be of simple type and limited variation. In particular choose the interval from

$$(0, 0, 0, \cdots) \text{ to } (b_1, b_2, \cdots, b_n, \cdots)$$

and let

$$\alpha(x_1, x_2, \cdots, x_n, \cdots) = \sum_{n=1}^{\infty} \frac{1}{n_4} \left[\prod_{s=1}^{n} \varphi(x_s) \right],$$

$\varphi(x) =$ integral part of $x \div (1 + x^2)$. $\varphi(x)$ is of limited variation from 0 to b_s, in fact from 0 to ∞, for, at $x = n$, it makes a sudden increase from

$$\frac{n-1}{1+n^2} \quad \text{to} \quad \frac{n}{1+n^2},$$

and then decreases continually from

$$\frac{n}{1+n^2} \quad \text{to} \quad \frac{n}{1+(n+1)^2},$$

when it suddenly increases again and so on. The variation of $\varphi(x)$ from 0 to ∞ is

$$V = \sum_{n=1}^{\infty} \frac{1}{1+n^2} + \frac{2n^2+n}{(n^2+2n+2)(n^2+1)} < 3\Sigma \frac{1}{1+n^2}$$

and therefore is finite.

The variation of α in the variables $(x_1, \cdots, x_n)$ or

$$\alpha_n < V^n \left(\frac{1}{n!} + \frac{1}{(n+1)!} + \cdots \right)$$

for $| \varphi(x) | < 1$.

$$\alpha_n < \frac{V^n}{(n-1)!}.$$

The right-hand expression has a finite upper bound, which proves that α is of limited variation, though, since this is only a particular case of the general example, we can see that the total variation is less than

$$\sum_{n=1}^{\infty} \frac{1}{n!} V^n = e^V - 1.$$

Example 2. Of an entirely different type is the following example. Let (x_n) denote the lower bound of the variables $(x_1, x_2, \cdots, x_n, \cdots)$ and let the interval be from $(0, 0, 0, \cdots)$ to $(3, 3, 3, \cdots)$; then

$$\begin{aligned} \alpha(x_1, \cdots, x_n, \cdots) &= 0 \qquad 0 \leq (x_n) < 1 \\ &= 1 \qquad 1 \leq (x_n) < 2 \\ &= -1 \qquad (x_n) \geq 2, \end{aligned}$$

is of limited variation and simple type. To prove this it is sufficient to observe that we can express α as the difference of two limited functions of simple and positive type, in fact,

$$\alpha = \beta - 2\beta'',$$

where

$$\begin{aligned}\beta(x_1, \cdots, x_n, \cdots) &= 0, \qquad 0 \leqq (x_n) < 1,\\ &= 1, \qquad (x_n) \geqq 1.\\ \beta''(x_1, \cdots, x_n, \cdots) &= 0, \qquad 0 \leqq (x_n) < 2,\\ &= 1, \qquad (x_n) \geqq 2.\end{aligned}$$

Consider the nth order difference of β in an interval

$$(c_1, c_2, \cdots, c_n) \text{ to } (x_1, x_2, \cdots, x_n)$$

the variables $x_{n+1}, x_{n+2}, \cdots$ being held constant. Since $c_r \leqq x_r$, if any $x_r < 1$, $\beta = 0$ at every vertex of the "interval" and the nth order difference is also 0. We suppose then that $(x_n) \geqq 1$. Let s of the c's be $\geqq 1$, the rest being < 1. Then at the extreme upper vertex $\beta = 1$, at s of the vertices next to this $\beta = 1$ and at the others $\beta = 0$, at $\binom{s}{2}$ of the next vertices $\beta = 1$ and at the others $\beta = 0$, and so on. Then the interval difference

$$\begin{aligned}\Delta^n\beta &= 1 - \binom{s}{1} + \binom{s}{2} - \cdots + (-1)^s\\ &= (1-1)^s\\ &= 0.\end{aligned}$$

If every $c_r < 1$, $\beta = 0$ at every vertex except the extreme upper vertex, at which $\beta = 1$. Hence the interval difference of β is always zero except when $(x_n) \geqq 1$ and $c_r < 1$, $(r = 1, \cdots, n)$, and in that case the interval difference $= 1$.

In particular, β is of positive type. Similarly it can be shown that β'' is of positive type.

It also follows that, if $f(x_p, \cdots, x_t)$ is a function of class T_0,

$$\int f d\beta = f(1, 1, \cdots, 1)$$

$$\int f d\beta'' = f(2, 2, \cdots, 2),$$

so that

$$\int f d\alpha = f(1, 1, \cdots, 1) - 2f(2, 2, \cdots, 2).$$

Lastly if f is any continuous function of the variables $(x_1, \cdots, x_n, \cdots)$,

$$\int f d\alpha = f(1, 1, \cdots, 1, \cdots) - 2f(2, 2, \cdots, 2, \cdots).$$

The reader can readily invent other examples of this type.

RICE INSTITUTE,
HOUSTON, TEXAS.